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On the Centralizers of Involutions in Finite Groups, II

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1. INTRODUCTION

As a tool for investigating the structure of the centralizers of involutions in a finite group G, we have introduced in [4] and [5], the concept of an *A-signalizer functor θ on G* for any abelian 2-subgroup A of G with $m(A) \geqslant 3$ and the related notion of the elements of $\mathcal{H}_\theta(A)$ in G and have there proved the following theorem: *If $m(A) \geqslant 6$ and G possesses the A-signalizer functor θ, then the elements of $\mathcal{H}_\theta(A)$ generate a subgroup of G of odd order.*[1]

Thus the theorem is applicable only in groups G whose *2-rank* is at least 6, where, by the *2-rank* of G is meant the maximum rank of an abelian 2-subgroup of G. This is a severe limitation, particularly since several of the families of presently known simple groups as well as certain of the sporadic simple groups have 2-rank less than 6. It is, therefore, clearly desirable to establish analogous results when the 2-rank of G is smaller than 6.

As pointed out in Section 6 of [4], the above-quoted result will also hold in the case $m(A) = 5$ provided one establishes an appropriate extension of Theorem 4.2 of [4]. In Section 2, we give a very simple proof of this extension, which will thus yield:

THEOREM A. *If A is an abelian 2-subgroup of the group G with $m(A) \geqslant 5$ and G possesses the A-signalizer functor θ, then the elements of $\mathcal{H}_\theta(A)$ in G generate a subgroup of G of odd order.*

However, it does not seem possible to extend all the results of [4] to obtain the corresponding theorem when $m(A) < 5$. The main purpose of the present paper is to show that an analogous theorem does, nevertheless, hold

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[1] We assume the reader is familiar with the various concepts and notation introduced in [4] and [5].

in the case $m(A) = 4$ provided θ is assumed to satisfy a certain additional condition. This condition was suggested, in part, by a result of Thompson's in his study of groups of order relatively prime to 3 and, in part, by the structure of the presently known simple groups of 2-rank at most 4.

In order to state this condition, we recall from [4] that if B is a noncyclic subgroup of A, then $\mathcal{U}_\theta(B)$ is the set of B-invariant subgroups K of G of odd order such that

$$K = \langle K \cap \theta(C(b)) \mid b \in B^\# \rangle, \tag{1}$$

and $\mathcal{U}_\theta(B; p)$, p an odd prime, is the set of elements of $\mathcal{U}_\theta(B)$ of order a power of p.

Definition. Let A be an abelian 2-subgroup of the group G with $m(A) \geqslant 3$ and let θ be an A-signalizer functor on G. We shall say that θ is *strongly flat* provided for each proper subgroup H of G such that $B = A \cap H$ is noncyclic, either the elements of $\mathcal{U}_\theta(B)$ contained in H generate a subgroup of H of odd order or else the following conditions are satisfied:

(a) B is a four group.

(b) Any two elements of $\mathcal{U}_\theta(B; p)$, p an odd prime, which are maximal subject to being contained in H, are conjugate by an element of $N_H(B)$.

(c) If K_i, $1 \leqslant i \leqslant m$, are the distinct elements of $\mathcal{U}_\theta(B)$ which are maximal subject to being contained in H, then

(1) $K_i \cap K_j / K_i \cap K_j \cap O(H)$ is cyclic for all $i \neq j$, and

(2) $K_i \cap K_k = K_j \cap K_k$ for all $i \neq j \neq k \neq i$.

In practice, if the elements of $\mathcal{U}_\theta(B)$ contained in H do not generate a subgroup of H of odd order and if L denotes the normal closure of B in H, then $L/O(L)$ will be isomorphic to either $PSL(2, q)$, $PGL(2, q)$, q odd, A_7, or to the extension of $PSL(3, 4)$ by an element of order 2 which is induced by the product of a field automorphism of order 2 of $SL(3, 4)$ with the transpose-inverse map of $SL(3, 4)$. In each of these cases, it is quite easy to verify that conditions (a), (b), and (c) are satisfied.

Actually we require this condition only in the case that H is an r-local subgroup of G for some odd prime r. Moreover, we remark that to show for a given subgroup H that the elements of $\mathcal{U}_\theta(B)$ contained in H generate a subgroup of odd order, it is enough to verify the following condition:

$$X = \langle O(C_H(b)) \mid b \in B^\# \rangle \text{ is of odd order.} \tag{2}$$

Indeed, suppose (2) holds and let $K \in \mathcal{U}_\theta(B)$ with $K \subseteq H$. Since

$$\theta(C(b)) \subseteq O(C(b))$$

by definition of an A-signalizer functor, it follows from Eq. (1) above that

$$K \subseteq \langle H \cap O(C(b)) \mid b \in B^{\#} \rangle.$$

But $H \cap O(C(b)) \subseteq O(C_H(b))$ and so $K \subseteq X$. Since this holds for each such element K and since X is of odd order by assumption, we see that the elements of $\text{И}_\theta(B)$ contained in H do, in fact, generate a subgroup of H of odd order.

We can now state our main result:

THEOREM B. *If A is an elementary abelian 2-subgroup of the group G with $m(A) = 4$ and G possesses the strongly flat A-signalizer functor θ, then the elements of $\text{И}_\theta(A)$ in G generate a subgroup of G of odd order.*

The assumption that A is elementary is not essential, but is made solely to simplify the notation.

In a subsequent paper with K. Harada, we shall use Theorem B to prove that Janko's two recently discovered simple groups of orders 604,800 and 50,232,960 are the only simple groups which have such a Sylow 2-subgroup of order 2^7. It is very likely that the theorem can be similarly used in studying simple groups of 2-rank 4 with other Sylow 2-subgroups.

The proof of Theorem B depends upon a variation of the results of Section 4 of [4]. The essential step is a proof of the following assertion: *If $P \in \text{И}_\theta^*(A; p)$, $P \neq 1$, p an odd prime, then for any odd prime $q \neq p$, either $C_P(Q) = 1$ for any element Q of $\text{И}_\theta^*(AP; q)$ or else there exists a nontrivial A-invariant subgroup Z of $Z(P)$ such that Z centralizes every element of $\text{И}_\theta(AZ; q)$.* Once this result is established, it is very easy to modify the proofs of the E- and C-theorems of Sections 5 and 6 of [4] to obtain the same conclusions. Theorem B then follows exactly as did Theorem A of [4].

Finally a word about groups of 2-rank less than 4. If G has 2-rank 0, it is of odd order and so is solvable. If G has 2-rank 1, a Sylow 2-subgroup of G is either cyclic or generalized quaternion and so G is not simple. If G has 2-rank 2 and G is simple, Alperin has shown in an unpublished work that a Sylow 2-subgroup of G is either dihedral, quasidihedral, wreathed (that is, the wreathed product of Z_{2^n}, $n \geqslant 2$, with Z_2) or else has order 2^6 and is isomorphic to a Sylow 2-subgroup of $U_3(4)$. Simple groups with such Sylow 2-subgroups have been subject to considerable investigation and have been largely, but not completely, classified.

This leaves the simple groups G of 2-rank 3. A considerable portion of the proof of Theorem B can be extended to this case provided one assumes in addition, that $N(A)/C(A)$ has order divisible by 7, A being an elementary abelian subgroup of G of order 8. This condition is, in fact, satisfied by each of the presently known simple groups of 2-rank 3 with the exception of M_{12}, which has already been classified by Brauer and Fong by its Sylow 2-subgroup and number of conjugacy classes of involutions. Thus one may reasonably hope that an effective analogue of Theorem B exists which can be used in the study of simple groups of 2-rank 3.

It will be convenient to adopt the "bar" convention: If $\bar{H}$ is a homomorphic image of the group H and X is a subgroup, subset, or element of H, $\bar{X}$ will always denote the image of X in $\bar{H}$.

2. Proof of Theorem A

Let A be an abelian 2-subgroup of the group G with $m(A) \geqslant 5$ and let θ be an A-signalizer functor on G. By the main results of [5], it suffices to prove the theorem under the additional assumption that θ is weakly flat. Thus if H is a p-lcoal subgroup of G, p odd, containing A, we can assume that the elements of $\mathcal{N}_\theta(A)$ contained in H generate a subgroup of H of odd order, which as usual we denote by $\theta(H)$. Furthermore, by the final remark of Section 6 of [4], it will be enough to demonstrate that Theorem 4.2 of [4] continues to hold under the weaker assumption $m(B) \geqslant 3$. Thus if $P \in \mathcal{N}_\theta^*(A; p)$, Z is a minimal A-invariant subgroup of $Z(P)$ and B is an elementary abelian subgroup of $C_A(Z)$ with $m(B) \geqslant 3$, we must establish the following assertion:

Theorem 2.1. *If* $K \in \mathcal{N}_\theta(BZ)$ *and* $[K, Z] = K$, *then*

$$C_K(Z) \subseteq O_{p'}(\theta(C(Z))).$$

Proof. Applying Lemma 2.4 of [4], it will once again suffice to prove the following statement: If X is an element of $\mathcal{N}_\theta(BZ)$ such that $[X, Z] = X$ and $m(B/C_B(X)) \leqslant 2$, then $Y = C_X(Z) \subseteq O_{p'}(\theta(C(Z)))$.

Since $m(B) \geqslant 3$, we have that $m(C_B(X)) \geqslant 1$ and, hence, that $C_B(X) \neq 1$. Let $b \in C_B(X)^\#$ and set $H = C(b)$. Then $\langle A, Z, X\rangle \subseteq H$ and Lemmas 3.3 and 4.1 of [4] imply that $C_P(b)$ is a Sylow p subgroup of $\theta(H)$ and that $X \subseteq \theta(H)$. We note that Lemma 4.1 of [4] actually holds for any noncyclic subgroup B of A. Since $Z \subseteq Z(C_P(b))$ and $[X, Z] = X$, it follows that $Z \subseteq O_{p',p}(\theta(H))$ and that $X \subseteq O_{p'}(\theta(H))$. Hence, if we set $K^* = [O_{p'}(\theta(H)), Z]$, we see that $X \subseteq K^*$ and that $Y \subseteq C_{K^*}(Z)$. Thus we need only prove that $C_{K^*}(Z) \subseteq O_{p'}(\theta(C(Z)))$.

By hypothesis, $m(A) \geqslant 5$ and consequently $C_A(Z)$ contains an elementary abelian subgroup B^* with $m(B^*) \geqslant 4$. Clearly K^* is B^* invariant and so $K^* \in \mathcal{N}_\theta(B^*)$. Furthermore, $[K^*, Z] = K^*$. Thus K^* and B^* satisfy the conditions of Theorem 4.2 of [4] and now the desired conclusion follows from that theorem.

3. Preliminary results

We now begin the proof of Theorem B. For the balance of the paper, A will denote an elementary abelian 2-subgroup of the group G with $m(A) = 4$ and θ will denote a strongly flat A-signalizer functor on G. We have already defined in Section 1 the set $И_\theta(B)$ for any noncyclic subgroup B of A. As in [4] we let $И_\theta(B; p)$ denote the set of elements of $И_\theta(B)$ of order a power of p, p an odd prime, and $И_\theta^*(B; p)$ the set of maximal elements of $И_\theta(B; p)$. Similarly if $P \in И_\theta(B; p)$, we denote by $И_\theta(BP; q)$ the set of P-invariant elements of $И_\theta(B; q)$, q any odd prime distinct from p, with a corresponding meaning for $И_\theta^*(BP; q)$.

If H is a proper subgroup of G such that $m(A \cap H) \geqslant 3$, then by the definition of strong flatness, the elements of $И_\theta(A \cap H)$ contained in H generate a subgroup of H of odd order. Following the notation of [4] and [5], we shall denote this subgroup by $\theta(H)$. Likewise, if $m(A \cap H) = 2$ and *if* the elements of $И_\theta(A \cap H)$ contained in H generate a subgroup of H of odd order (which according to the definition of strong flatness need not be the case), we denote this subgroup by $\theta(H)$.

For the sake of clarity, we repeat here the statements of Theorem 2.6, Lemma 2.7, Theorems 3.1 and 3.2, and Lemmas 3.3 and 4.1 of [4], all of which hold for any abelian 2-subgroup A with $m(A) \geqslant 3$ and any A-signalizer functor θ on G.

Theorem 3.1. *If $K \in И_\theta(A)$, then $C_K(a) = K \cap \theta(C(a))$ for each a in $A^\#$.*

Lemma 3.2. *If B is a noncyclic subgroup of A and $K \in И_\theta(A)$, then $K \in И_\theta(B)$.*

Theorem 3.3. *For any odd prime p, $\theta(C(A))$ permutes the elements of $И_\theta^*(A; p)$ transitively under conjugation.*

Theorem 3.4. *If $P \in И_\theta^*(A; p)$, p an odd prime, then*

$$N(A) = (N(A) \cap N(P))\,\theta(C(A)).$$

Lemma 3.5. *If $P \in И_\theta^*(A; p)$, p an odd prime, and $a \in A^\#$, then $C_P(a)$ is a Sylow p-subgroup of $\theta(C(a))$.*

In particular, we see that P contains a Sylow p-subgroup of $\theta(C(a))$ for each a in $A^\#$.

Lemma 3.6. *If H is a proper subgroup of G containing A and B is a noncyclic subgroup of A, then every element of $И_\theta(B)$ contained in H lies in $\theta(H)$.*

These results, in turn, have a number of consequences, the first of which is an extension of Lemmas 3.2 and 3.6.

LEMMA 3.7. *If A_0 is a subgroup of A with $m(A_0) \geqslant 3$ and B is a noncyclic subgroup of A_0, then we have:*

(i) *Every element of $\mathcal{M}_\theta(A_0)$ is contained in $\mathcal{M}_\theta(B)$.*

(ii) *If H is a proper subgroup of G such that $A_0 = A \cap H$, then every element of $\mathcal{M}_\theta(B)$ contained in H lies in $\theta(H)$.*

Proof. If $B = A_0$, (i) is obvious and (ii) is immediate from the definition of $\theta(H)$. Likewise if $A = A_0$, (i) follows from Lemma 3.2 and (ii) from Lemma 3.6. Hence, it remains to consider the case $m(B) = 2$, $m(A_0) = 3$, and $m(A) = 4$.

In this case, we define an A_0-signalizer functor θ_0 on G by setting, for each a in $A_0^{\#}$,

$$\theta_0(C(a)) = \theta(C(a)).$$

That θ_0 is an A_0-signalizer functor on G is easily verified. Indeed, conditions (a) and (b) of the definition follow at once for θ_0 from the corresponding conditions for θ. Moreover, if B_0 is a subgroup of A_0 with $m(A_0/B_0) \leqslant 2$, condition (c) will follow for θ_0 and B_0 from the corresponding condition for θ provided also $m(A/B_0) \leqslant 2$. In the contrary case, $m(A/B_0) = 3$ and B_0 is of order 2, in which case condition (c) is a direct consequence of the fact that $\theta_0(C(a))$ is normal in $C(a)$ by condition (a).

Note also that $\mathcal{M}_\theta(A_0)$ and $\mathcal{M}_{\theta_0}(A_0)$ consist of the same subgroups of G, as do $\mathcal{M}_\theta(B)$ and $\mathcal{M}_{\theta_0}(B)$. Hence, (i) follows at once from Lemma 3.2 applied to the A_0-signalizer functor θ_0. Furthermore, if H is as in (ii), it follows that $\theta(H) = \theta_0(H)$, where $\theta_0(H)$ has the obvious meaning. We can, therefore, apply Lemma 3.6 to A_0 and θ_0 to conclude that every element of $\mathcal{M}_{\theta_0}(B)$ contained in H lies in $\theta_0(H)$, and (ii) follows.

We now fix an odd prime p and an element P of $\mathcal{M}_\theta^*(A; p)$ and derive a number of results concerning P. First, the argument that established Lemma 3.5 can be repeated with obvious modifications with any nontrivial subgroup B of A in place of the element a and yields

LEMMA 3.8. *If B is a nontrivial subgroup of A, then $C_p(B)$ is a Sylow p-subgroup of $\theta(C(B))$.*

Combining Lemmas 3.2 and 3.5, we also have

LEMMA 3.9. *If B is a noncyclic subgroup of A, then $P \in \mathcal{M}_\theta^*(B; p)$.*

Proof. By Lemmas 3.2 and 3.5, $P \in \mathcal{M}_\theta(B; p)$ and $C_p(b)$ is a Sylow p-subgroup of $\theta(C(b))$ for each b in $B^{\#}$. But now if Q is an element of

$И_\theta^*(B; p)$ containing P, then $Q = \langle Q \cap \theta(C(b)) \mid b \in B^\#\rangle$ and $Q \cap \theta(C(b))$ is a p-subgroup of $\theta(C(b))$ containing $P \cap \theta(C(b)) = C_P(b)$ for each b in $B^\#$. Since $C_P(b)$ is a Sylow p-subgroup of $\theta(C(b))$, it follows that $Q \cap \theta(C(b)) = C_P(b) \subseteq P$ and consequently $Q \subseteq P$. Thus $Q = P$ and the lemma is proved.

Using this last result together with the fact that θ is strongly flat, we obtain the following important transitivity theorem.

THEOREM 3.10. *If B is a noncyclic subgroup of A, then $N(B)$ permutes the elements of $И_\theta^*(B; p)$ transitively under conjugation.*

Proof. Let $P \in И_\theta^*(A; p)$. Lemmas 3.5 and 3.9 imply that $P \in И_\theta^*(B; p)$ and that $C_P(\mathrm{b})$ is a Sylow p-subgroup of $\theta(C(b))$ for each b in $B^\#$. We shall argue that any element Q of $И_\theta^*(B; p)$ is conjugate to P by an element of $N(B)$. Assume false and choose Q to violate the conclusion with $D = P \cap Q$ of maximal order. We can clearly assume that $1 \notin И_\theta^*(B; p)$, in which case $Q \neq 1$ and consequently $C_Q(b) \neq 1$ for some b in $B^\#$. But $(C_Q(b))^x \subseteq C_P(b)$ for some element x in $C_{\theta(C(b))}(B)$ as $C_Q(b)$ is a B-invariant p-subgroup of $\theta(C(b))$ and $C_P(b)$ is a B-invariant Sylow p-subgroup of $\theta(C(b))$. Thus $Q^x \cap P \neq 1$. Since Q is not conjugate to P by an element of $N(B)$, neither is Q^x and we conclude at once from our maximal choice of Q that $D \neq 1$. Clearly $D \subset P$ and $D \subset Q$.

Setting $H = N(D)$, we have $H \cap P = N_P(D) \supset D$ and $H \cap Q = N_Q(D) \supset D$. We let U, V be elements of $И_\theta^*(B; p)$ containing $N_P(D)$, $N_Q(D)$, respectively, and such that $U \cap H$, $V \cap H$ are of maximal order. This last condition implies that $U \cap H$ and $V \cap H$ are elements of $И_\theta(B; p)$ which are maximal subject to being contained in H. If the elements of $И_\theta(B)$ contained in H generate a subgroup K of H of odd order, then clearly $U \cap H$ and $V \cap H$ are, in fact, B-invariant Sylow p-subgroups of K and so $(U \cap H)^y = V \cap H$ for some element y in $C(B) \subseteq N(B)$. On the other hand, if the elements of $И_\theta(B)$ contained in H do not generate a subgroup of H of odd order, it follows directly from the definition of strong flatness that $(U \cap H)^y = V \cap H$ for some element y in $N_H(B) \subseteq N(B)$. In either case, we reach a contradiction by a now standard argument.

We next prove

LEMMA 3.11. *If $SCN_3(P)$ is empty, then $C_A(P)$ is noncyclic.*

Proof. Assume $SCN_3(P)$ is empty. Let C be a critical subgroup of P and set $D = \Omega_1(C)$. Then by Theorem 5.3.11 and Lemma 5.3.9 of [3], $C_A(D) = C_A(P)$ and D is of class at most 2 and of exponent p. Since $SCN_3(P)$ is empty and p is odd, every abelian subgroup of P can be generated by at most 2 elements by Theorem 5.4.15 of [3]. In particular, this is true of D and we conclude easily from the structure of D that D is either elementary abelian of order at most p^2 or extra-special of order p^3. Setting $\bar{D} = D/\Phi(D)$, it follows in either case that $Aut(\bar{D})$ is isomorphic to a subgroup of $GL(2, p)$. But $GL(2, p)$ has 2-rank 2 and consequently $m(A/C_A(\bar{D})) \leqslant 2$. Since $C_A(\bar{D}) = C_A(D)$ by a theorem of Burnside (Theorem 5.1.4 of [3]), we conclude that $C_A(D)$ is noncyclic. Since $C_A(D) = C_A(P)$, the lemma follows.

To obtain results about the action of A on P when $SCN_3(P)$ is nonempty, we make use of the following basic fact:

LEMMA 3.12. *The group AP is supersolvable.*

Proof. Since A is an elementary abelian 2-group, the lemma follows at once from the fact that the irreducible representations of such a group A on a vector space over $GF(p)$ are necessarily one-dimensional.

By repeated use of the supersolvability of AP, we obtain at once the following corollary, the details of which are left to the reader:

LEMMA 3.13. *If $SCN_3(P)$ is nonempty, then the following conditions hold:*

(i) *$\Omega_1(Z(P))$ contains an A-invariant subgroup of order p.*

(ii) *If Z is an A-invariant subgroup of $\Omega_1(Z(P))$ of order p, then Z is contained in an A-invariant element of $U(P)$.*

(iii) *If U is an A-invariant element of $U(P)$, then U is contained in an A-invariant elementary abelian normal subgroup E of P of order p^3.*

(iv) *If E is an A-invariant normal elementary abelian subgroup of P, then E is contained in an A-invariant element of $SCN_3(P)$.*

In the following lemma, $C_A^*(X)$ will denote the set of elements of A which either centralize or invert the subgroup X of P.

LEMMA 3.14. *If $SCN_3(P)$ is nonempty, then the following conditions hold:*

(i) *If Z is an A-invariant subgroup of $Z(P)$ of order p, then $m(C_A(Z)) \geqslant 3$.*

(ii) *If U is an A-invariant element of $U(P)$, then*

(a) $m(C_A(U)) \geqslant 2$;

(b) $m(C_A^*(U)) \geqslant 3$.

(c) *If the involution a of A centralizes or inverts U, then a centralizes $P/C_P(U)$.*

(iii) *If E is an A-invariant elementary abelian subgroup of P of order p^3, then*

(a) $m(C_A(E)) \geqslant 1$;

(b) $m(C_A^*(E)) \geqslant 2$.

Proof. First, (i) is an immediate consequence of the fact that $Aut(Z)$ is cyclic as $m(A) = 4$. Since $Aut(U)$ is isomorphic to $GL(2, p)$ which has 2-rank 2, (ii) (a) also follows. Clearly in proving (ii) (b), we can assume that

$m(C_A(U)) = 2$. Hence, if B is a complement of $C_A(U)$ in A, B is a four group and acts faithfully on U. But $GL(2, p)/SL(2, p)$ is cyclic and $SL(2, p)$ possesses a unique involution y. Identifying $GL(2, p)$ with $Aut(U)$, we have that y inverts U. Furthermore, we see that any noncyclic abelian 2-subgroup of $Aut(U)$ necessarily contains y, so $y \in B$. Thus $C_A^*(U) = \langle C_A(U), y\rangle$ and so $m(C_A^*(U)) \geqslant 3$, proving (ii)(b).

Similarly $Aut(E)$ is isomorphic to $GL(3, p)$. One checks that $GL(3, p)$ has 2-rank 3 and that if B is an elementary abelian subgroup of A of order 8 which is faithfully represented on E, then some involution of B must invert E. But now both parts of (iii) follow in the same way as (ii)(a) and (ii)(b).

We turn now to (ii)(c), which we suppose false for some involution a of A which either centralizes or inverts U. Setting $\bar{P} = P/C_P(U)$, we clearly have that $\bar{P} \neq 1$. Since $U \in U(P)$, $\bar{P}$ has order p. Since a does not centralize $\bar{P}$, it necessarily inverts $\bar{P}$. This implies that a inverts some element x of $P - C_P(U)$. We can write $U = \langle z, u\rangle$, where $z \in Z(P)$ and

$$[u, x] = z.$$

Conjugation by a yields

$$[u^a, x^{-1}] = z^a.$$

The latter relation implies, whether a centralizes or inverts U, that $[u, x] = z^{-1}$. Hence, $z = z^{-1}$ and $z^2 = 1$. Since z has odd order, this forces $z = 1$, which is not the case. Thus (ii)(c) also holds.

We also need

LEMMA 3.15. *Let U be an A-invariant element of $U(P)$, F an A-invariant element of $SCN_3(P)$ containing U, and B a subgroup of $C_A^*(U)$. If BU or BF is irreducibly represented on a vector space V over $GF(q)$, q an odd prime distinct from p, then U does not act faithfully on V.*

Proof. This is essentially a direct consequence of the supersolvability of BU and BF, but we prove it for completeness. We apply Clifford's theorem (Theorem 5.4.1 of [3]) to the normal subgroup U of BU or BF. Let W be a corresponding Wedderburn component of V. Since U is abelian of type (p, p) and W is the direct sum of isomorphic U-modules, we have $U_0 = C_U(W) \neq 1$. However, as F is abelian, every element of BU or BF, as the case may be, either centralizes or inverts U, so U_0 is normal in BU or BF. But every Wedderburn component of V is conjugate to W by an element of BU or BF, as the case may be. We see then that U_0 centralizes each Wedderburn component of V and, therefore, U_0 centralizes V. Thus U is not represented faithfully on V.

Our concluding result is basic for the paper:

THEOREM 3.16. *Let $P \in \mathcal{H}_\theta^*(A; p)$, p an odd prime, let B be a four subgroup of A, and let Z be a subgroup of $C_{Z(P)}(B)$. Then if K is an element of $\mathcal{H}_\theta(B)$ containing Z, we have that $Z \subseteq O_{p',p}(K)$.*

Proof. Let b_i, $1 \leqslant i \leqslant 3$, denote the involutions of B and let R be a BZ-invariant Sylow p-subgroup of $O_{p',p}(K)$. Setting $R_i = C_R(b_i)$, $1 \leqslant i \leqslant 3$, we have $R = R_1R_2R_3$ by Lemma 10.5.1 of [3]. It will suffice to prove that $[R_i, Z, Z] = 1$, $1 \leqslant i \leqslant 3$. Indeed, assume this to be the case and set $\bar{K} = K/O_{p'}(K)\,\Phi(R)$. Since K is solvable, being of odd order, it is p constrained and so $C_K(R) \subseteq O_{p'}(K)R$. It follows, therefore, from a theorem of Burnside (Theorem 5.1.4 of [3]) together with the fact that $C_{\bar{K}}(\bar{R})$ is the image of $C_K(R)$ that $C_{\bar{K}}(\bar{R}) \subseteq \bar{R}$, whence $O_{p'}(\bar{K}) = 1$ and $\bar{R} = O_p(\bar{K})$. Furthermore, $\bar{R} = \bar{R}_1\bar{R}_2\bar{R}_3$ is elementary abelian and by assumption, $[\bar{R}_i, \bar{Z}, \bar{Z}] = 1$. Since $\bar{R}$ is abelian, the latter condition implies that $[\bar{R}, \bar{Z}, \bar{Z}] = 1$. But $\bar{K}$ is p-stable by Theorem 2.8.4 of [3] as it has odd order and so $\bar{Z} \subseteq C_{\bar{K}}(\bar{R}) \subseteq \bar{R}$ by the definition of p-stability. We conclude that $Z \subseteq O_{p',p}(K)$, as required.

To prove the desired assertion, set $C_i = C(b_i)$, $K_i = \theta(C_i)$, and $P_i = C_P(b_i)$, $1 \leqslant i \leqslant 3$. By Lemma 3.5, P_i is a Sylow p subgroup of K_i. Furthermore, $R_i \subseteq C_i$ and $R_i \in \mathcal{H}_\theta(B)$. Since $A \subseteq C_i$, Lemma 3.6 implies that $R_i \subseteq K_i$, $1 \leqslant i \leqslant 3$. Now set $\bar{K}_i = K_i/O_{p'}(K_i)$ and $\bar{Q}_i = O_p(\bar{K}_i)$. Then $\bar{P}_i$ is a Sylow p subgroup of $\bar{K}_i$ and $\bar{Q}_i \subseteq \bar{P}_i$. Since $\bar{K}_i$ is p constrained, $C_{\bar{K}_i}(\bar{Q}_i) \subseteq \bar{Q}_i$. However, $Z \subseteq P_i$ as Z centralizes B and so $Z \subseteq Z(P_i)$ as $Z \subseteq Z(P)$. Hence, $\bar{Z} \subseteq Z(\bar{P}_i)$ and so $\bar{Z}$ centralizes $\bar{Q}_i$. It follows, therefore, that $\bar{Z} \subseteq Z(\bar{Q}_i)$. But $\bar{R}_i$ normalizes $\bar{Q}_i$ and so $\bar{R}_i$ also normalizes $Z(\bar{Q}_i)$. Hence, $[\bar{R}_i, \bar{Z}] \subseteq Z(\bar{Q}_i)$ and, therefore, $[\bar{R}_i, \bar{Z}, \bar{Z}] = 1$. We conclude at once that $[R_i, Z, Z] = 1$, $1 \leqslant i \leqslant 3$, as required.

As a corollary we have

LEMMA 3.17. *Under the assumptions of the theorem, if Q is a Z-invariant p' subgroup of K such that $[Q, Z] = Q$, then $Q \subseteq O_{p'}(K)$.*

Proof. Since $Z \subseteq O_{p',p}(K)$ by the theorem, we have

$$[Q, Z] \subseteq O_{p',p}(K) \cap Q \subseteq O_{p'}(K).$$

4. A RELATIVIZED TRANSITIVITY THEOREM AND SOME CONSEQUENCES

In this section we consider an element P of $\mathcal{H}_\theta^*(A; p)$, p an odd prime, for which $C_A(P)$ is cyclic (possibly trivial). By Lemma 3.11, $SCN_3(P)$ is necessarily nonempty under these conditions. We let U be an A-invariant element of $U(P)$ and let F be an A-invariant element of $SCN_3(P)$ containing U. These exist by Lemma 3.13. Furthermore, by Lemmas 3.13(iv) and 3.14(iii) there exists an A-invariant subgroup E of F of type (p, p, p) containing U such that $C_A^*(E)$ is noncyclic. We let B be a four subgroup of $C_A^*(E)$. Finally let Z be an A-invariant subgroup of $U \cap Z(P)$ of order p and fix all this notation.

Our goal will be to derive a "relativized" transitivity theorem for the elements of $\mathcal{V}_\theta^*(BF; q)$, q an odd prime distinct from p. We shall then apply this result to obtain information about the elements of $\mathcal{V}_\theta(B_0Z; q)$, where B_0 is a four subgroup of $C_A(Z)$ (B_0 not necessarily equal to B). We fix the prime q as well.

We begin with some preliminary results, the first of which is proved exactly as in Lemma 8.5.2 of [3] and is left to the reader.

LEMMA 4.1. *If K is an element of $\mathcal{V}_\theta(B)$ containing F such that $P \cap K$ is a Sylow p subgroup of K, then every element of $\mathcal{V}_\theta(BF; q)$ contained in K lies in $O_{p'}(K)$.*

If U_0 is a subgroup of U of order p and $H = N(U_0)$, then clearly $C_A^*(U) \subseteq H$ and consequently $m(A \cap H) \geqslant 3$ by Lemma 3.14(ii). Since θ is strongly flat, it follows that the elements of $\mathcal{V}_\theta(A \cap H)$ contained in H generate a subgroup $\theta(H)$ of odd order. Furthermore, $B \subseteq A \cap H$ as $B \subseteq C_A^*(E) \subseteq C_A^*(U)$. We conclude, therefore, from Lemma 3.7 that every element of $\mathcal{V}_\theta(B)$ contained in H lies in $\theta(H)$. We shall make repeated use of these facts.

We next prove

LEMMA 4.2. *Let Q be an element of $\mathcal{V}_\theta(BF; q)$ which centralizes a subgroup U_0 of U of order p with either $U_0 \subseteq Z(P)$ or $[Q, Z] = Q$. If $H = N(U_0)$, then $Q \subseteq O_{p'}(\theta(H))$.*

Proof. We have $QF \subseteq H$ and $QF \in \mathcal{V}_\theta(B)$, so $QF \subseteq \theta(H)$ by the preceding remarks. Likewise $P \cap H \subseteq \theta(H)$ and $| P : P \cap H | \leqslant p$ with $P \cap H = P$ if $U_0 \subseteq Z(P)$. Since $P \in \mathcal{V}_\theta^*(A; p)$, $P \in \mathcal{V}_\theta^*(B; p)$ by Lemma 3.9 and, thus, by Theorem 3.10 $P \cap H$ is of index at most p in a Sylow p subgroup of $\theta(H)$ with P itself a Sylow p-subgroup of $\theta(H)$ if $P \subseteq H$.

If $P \subseteq H$, the preceding lemma now yields that $Q \subseteq O_{p'}(\theta(H))$ inasmuch as $\theta(H) \in \mathcal{V}_\theta(B)$. Suppose, on the other hand, that $P \not\subseteq H$, in which case $U_0 \not\subseteq Z(P)$ and $[Q, Z] = Q$. Since $m(A \cap H) \geqslant 3$, $C_{A \cap H}(Z)$ contains a four subgroup B_0 by Lemma 3.14(i). Since also $\theta(H) \in \mathcal{V}_\theta(B_0)$, we conclude at once from Lemma 3.17 that $Q \subseteq O_{p'}(\theta(H))$ in this case as well.

This in turn yields

LEMMA 4.3. *If H is a proper subgroup of G containing BF, then there exists an element K of $\mathcal{V}_\theta(B)$, maximal subject to the conditions $F \subseteq K \subseteq H$, such that every element of $\mathcal{V}_\theta(BF; q)$ contained in H lies in $O_{p'}(K)$.*

Proof. Using the definition of strong flatness, we first argue that such an element K exists which contains every element of $\mathcal{U}_\theta(BF; q)$; so assume the contrary. Then there must exist distinct elements K_1, K_2 in $\mathcal{U}_\theta(B)$ which are maximal subject to the conditions $F \subseteq K_i \subseteq H$ and also elements Q_1, Q_2 of $\mathcal{U}_\theta(BF; q)$ with $Q_i \subseteq K_i$, but $Q_i \not\subseteq K_j$, $i \neq j$, $1 \leqslant i, j \leqslant 2$.

We set $K_0 = K_1 \cap K_2$, so that $F \subseteq K_0$; in particular, $U \subseteq K_0$. Furthermore, clearly K_1 and K_2 both contain every element of $\mathcal{U}_\theta(B)$ which lies in $O(H)$ and consequently $K_1 \cap O(H) = K_2 \cap O(H) = K_0 \cap O(H)$. Moreover, by definition of strong flatness, $K_0/K_0 \cap O(H)$ is cyclic; hence, $K_0 \cap O(H)$ contains a subgroup U_0 of U of order p. Since $K_i \cap O(H)$ is normal in K_i, it follows that $[Q_i, U_0] \subseteq K_i \cap O(H) = K_0 \cap O(H)$. But $Q_i = [Q_i, U_0]\, C_{Q_i}(U_0)$. Since $Q_i \not\subseteq K_j$ for $i \neq j$, we conclude, therefore, that $C_{Q_i}(U_0) \not\subseteq K_j$ for $i \neq j$, $1 \leqslant i, j \leqslant 2$.

We claim that $X = \langle C_{Q_1}(U_0), C_{Q_2}(U_0)\rangle$ is of even order. Indeed, in the contrary case, $X \in \mathcal{U}_\theta(B)$. But then if K_3 denotes an element of $\mathcal{U}_\theta(B)$, which is maximal subject to $X \subseteq K_3 \subseteq H$, our conditions imply that $K_3 \neq K_1$ or K_2 and that

$$K_1 \cap K_3 \neq K_2 \cap K_3,$$

contrary to the definition of strong flatness. This proves the assertion.

Since $C_{Q_i}(U_0) \in \mathcal{U}_\theta(B)$, it follows from the preceding argument that the elements of $\mathcal{U}_\theta(B)$ contained in $N(U_0)$ do not generate a subgroup of $N(U_0)$ of odd order, contrary to what we have shown above. This proves the existence of the element K of $\mathcal{U}_\theta(B)$ with the required properties.

Suppose now that there exists an element Q of $\mathcal{U}_\theta(BF; q)$ contained in K, but not contained in $O_{p'}(K)$. We choose Q minimal with these properties, in which case $\bar{Q} = Q/Q \cap O_{p'}(K)$ is elementary abelian and BF acts irreducibly on $\bar{Q}$. Since $B \subseteq C_A^*(U)$, Lemma 3.15 now yields that $\bar{Q}$ centralizes a subgroup U_1 of U of order p. If possible, we choose $U_1 \subseteq Z(P)$. Since $C_Q(U_1)$ maps onto $\bar{Q}$, the minimality of Q implies that U_1 centralizes Q. Moreover, if $U_1 \not\subseteq Z(P)$, then Z does not centralize $\bar{Q}$, whence $\bar{Q} = [\bar{Q}, Z]$ and $Q = [Q, Z]$, again by the minimality of Q. We conclude therefore from the preceding lemma that $Q \subseteq O_{p'}(\theta(N(U_1)))$.

Finally let R be a BU-invariant Sylow p subgroup of $O_{p',p}(K)$. Then $C_R(U_1) \subseteq \theta(N(U_1))$ and consequently $[Q, C_R(U_1)]$ is a p' group. But now setting $\bar{K} = K/O_{p'}(K)$ and using the fact that $C_{\bar{R}}(\bar{U}_1)$ is the image of $C_R(U_1)$ in $\bar{K}$ and that $\bar{R} = O_p(\bar{K})$, we conclude at once that $\bar{Q}$ centralizes $C_{\bar{R}}(\bar{H}_1)$. Application of Theorem 5.3.4 of [3] now yields that $\bar{Q}$ centralizes $\bar{R}$. But $\bar{K}$, being of odd order, is solvable and so $C_{\bar{K}}(\bar{R}) \subseteq \bar{R}$ as $\bar{K}$ is p constrained. This forces $\bar{Q} = 1$ and, hence, $Q \subseteq O_{p'}(K)$, completing the proof of the lemma.

We note that Lemma 4.3 is valid for every odd prime $q \neq p$. In particular, if K is an element of $\mathcal{U}_\theta(B)$ containing F, it follows from the lemma that every

BF-invariant p' subgroup of K lies in $O_{p'}(K)$. Using Lemma 4.3 together with this consequence of it, we can now establish our desired transitivity theorem.

THEOREM 4.4. *$O_{p'}(\theta(C(BF)))$ permutes the elements of $И_\theta^*(BF; q)$ transitively under conjugation.*

Proof. Observe that as $C(BF)$ is A-invariant, $\theta(C(BF))$ is well-defined. The proof of the theorem follows the standard pattern of the Thompson transitivity theorem (Theorem 8.5.4 of [3] or Theorem 17.1 of [1]). The critical point is to establish an appropriate preliminary assertion concerning $H = N(E_0)$ for any subgroup E_0 of E of order p: Namely, if Q_1 and Q_2 are any two elements of $И_\theta(BF; q)$ maximal subject to being contained in H, then $Q_2 = Q_1{}^x$ for some element x in $O_{p'}(\theta(C(BF))) \cap H$. We shall verify this one assertion, but shall leave the remaining details to the reader.

We apply the preceding lemma with $H = N(E_0)$ and let K be as in that lemma. By the lemma $Q_i \subseteq O_{p'}(K)$, $1 \leqslant i \leqslant 2$. However, our maximal choice of Q_i clearly implies that each Q_i is a Sylow q subgroup of $O_{p'}(K)$. Hence, $Q_2 = Q_1{}^x$ for some element x in $R = C(BF) \cap O_{p'}(K)$. Furthermore, $R \in И_\theta(B)$ and so R is a BF-invariant p' subgroup of $\theta(C(BF))F$. Since the latter group is an element of $И_\theta(B)$ containing F, we conclude at once from the argument preceding the theorem that $R \subseteq O_{p'}(\theta(C(BF)))$. Since $x \in R$ and $R \subseteq H$, the desired assertion follows.

Reasoning as in Theorems 4.5 and 4.6 of [4], we obtain the following important corollary:

THEOREM 4.5. *$AC_P(B)$ normalizes some element of $И_\theta^*(BF; q)$.*

We wish to prove that P normalizes some element of $И_\theta^*(BF; q)$. To do this, we require a preliminary result which will yield the desired conclusion in the case $U \subseteq Z(P)$.

LEMMA 4.6. *$C_P(U)$ normalizes some element of $И_\theta^*(BF; q)$.*

Proof. The lemma is obvious if $И_\theta^*(BF; q)$ is trivial, so we may assume the contrary. Set $P_0 = C_P(U)$ and $A_0 = C_A^*(U)$. We first argue that $И_\theta(A_0P_0; q)$ is nontrivial. Let Q be a minimal element of $И_\theta(BF; q)$, so that Q is elementary abelian and BF acts irreducibly on Q. Application of Lemma 3.15, as in Lemma 4.3, yields a subgroup U_0 of U of order p which centralizes Q with either $U_0 \subseteq Z(P)$ or $[Q, Z] = Q$. Setting $H = N(U_0)$, Lemma 4.2 implies that $Q \subseteq O_{p'}(\theta(H))$. But $A_0P_0 \subseteq H$; hence, $P_0 \subseteq \theta(H)$, so P_0 normalizes some A_0-invariant Sylow q subgroup Q_0 of $O_{p'}(\theta(H))$. Since $Q \subseteq O_{p'}(\theta(H))$ and $Q \neq 1$, also $Q_0 \neq 1$. Thus, Q_0 is a nontrivial element of $И_\theta(A_0P_0; q)$. In particular, it follows that $1 \notin И_\theta^*(A_0P_0; q)$.

Next let $Q_1 \in И_\theta^*(A_0P_0; q)$ and suppose that $Q_1 \notin И_\theta^*(BF; q)$. Since $BF \subseteq A_0P_0$, $Q_1 \in И_\theta(BF; q)$ and so there exists an element Q^* of $И_\theta^*(BF; q)$ with $Q_1 \subseteq Q^*$. By our assumption, $Q_1 \subset Q^*$. Now set $M = N(Q_1)$. Since $m(A_0) \geqslant 3$ by Lemma 3.14(ii), $\theta(M)$ as well-defined; hence, every element of $И_\theta(B)$ contained in M lies in $\theta(M)$ by Lemma 3.7. In particular, Lemma 4.3 applies to M with $K = \theta(M)$. Since $Q_2 = N_{Q^*}(Q_1) \in И_\theta(BF; q)$, we thus have that $Q_2 \subseteq O_{p'}(\theta(M))$. But $A_0 \subseteq M$, $P_0 \subseteq \theta(M)$, and $Q_2 \supset Q_1$. Hence, if Q_3 is an A_0P_0-invariant Sylow q subgroup of $O_{p'}(\theta(M))$ containing Q_1, then $Q_3 \supset Q_1$. Since $Q_3 \in И_\theta(A_0P_0; q)$, our maximal choice of Q_1 is thereby contradicted, and the lemma is proved.

We can now prove

THEOREM 4.7. *AP normalizes some element of $И_\theta^*(BF; q)$.*

Proof. We set $P_0 = C_P(U)$ and $P_1 = C_P(B)$. By our choice of B, every involution b of B either centralizes U or inverts U. In either case b centralizes $P/C_P(U)$ by Lemma 3.14(ii) and so B centralizes $P/C_P(U)$. Thus $P = P_0P_1$. Moreover, either $P = P_0$ or $P_0 \cap P_1$ is of index p in P_1.

Now by Theorem 4.5, AP_1 normalizes some element Q of $И_\theta^*(BF; q)$. Since the theorem holds trivially if $Q = 1$, we may assume that $Q \neq 1$. We set $H = N(Q)$, so that $AP_1 \subseteq H$. In particular, the elements of $И_\theta(A)$ contained in H generate a subgroup $\theta(H)$ of H of odd order and once again Lemma 3.7 implies that every element of $И_\theta(B)$ contained in H lies in $\theta(H)$. Furthermore, by Theorem 3.10 and Lemma 4.6, $P_0^x \subseteq H$ for some element x in $N(B)$. Since $m(A/B) = 2$, condition (c) in the definition of A-signalizer functor implies that $P_0^x \in И_\theta(B)$; thus $P_0^x \subseteq \theta(H)$. Likewise $P_1 \subseteq \theta(H)$.

We let R be an A-invariant Sylow p subgroup of $\theta(H)$ containing P_1. If $P_0 = P$, then $|R| \geqslant |P|$. But as $R \in И_\theta(A; p)$, it follows from Theorem 3.3 that $|R| = |P|$ and that $R \in И_\theta^*(A; p)$. We argue to the same conclusion when $P_0 \subset P$. Indeed, in this case $|P : P_0| = p$ and $|P_1 : P_0 \cap P_1| = p$. Moreover, $P_0^{xy} \subseteq R$ for some element y in $C_{\theta(H)}(B)$. Hence, by another application of Theorem 3.10, P_0^{xy} has index at most p in R and if the index is p, then $R \in И_\theta^*(A; p)$. Suppose then, by way of contradiction, that $P_0^{xy} = R$. Then $P_1 \subseteq P_0^{xy}$ and so $|C_{P_0^{xy}}(B)| \geqslant |P_1|$. However,

$$|C_{P_0^{xy}}(B)| = |C_{P_0}(B)| = |P_0 \cap P_1| \qquad \text{as} \quad xy \in N(B).$$

Thus, $|P_0 \cap P_1| \geqslant |P_1|$, contrary to the fact that $|P_1 : P_0 \cap P_1| = p$. We conclude that $R \in И_\theta^*(A; p)$ in this case as well.

Finally $P = R^u$ for some element u in $\theta(C(A))$ by Theorem 3.3 inasmuch as P and R are in $И_\theta^*(A; p)$. Thus AP normalizes Q^u as AR normalizes Q. But $Q^u \in И_\theta^*(BF; q)$ as u centralizes B and the theorem is proved.

As a direct corollary we have

THEOREM 4.8. *If P centralizes every element of $\mathcal{V}_\theta^*(AP; q)$, then F centralizes every element of $\mathcal{V}_\theta^*(BF; q)$.*

Proof. Let $Q \in \mathcal{V}_\theta^*(BF; q)$. By Theorem 4.7, AP normalizes some element Q_1 of $\mathcal{V}_\theta^*(BF; q)$ and by Theorem 4.5, $Q = Q_1^x$ with $x \in \theta(C(BF))$. By hypothesis, P centralizes Q_1. In particular, F centralizes Q_1. Since x centralizes F, we conclude that F centralizes Q and the theorem is proved.

We can now establish the result which is the key conclusion we need from this analysis for the proof of Theorem B. Note that by Lemma 3.14(i), $m(C_A(Z)) \geqslant 3$.

THEOREM 4.9. *If P centralizes every element of $\mathcal{V}_\theta^*(AP; q)$ and B_0 is a four subgroup of $C_A(Z)$, then Z centralizes every element of $\mathcal{V}_\theta^*(B_0Z; q)$.*

Proof. Suppose false and choose Q_0 of minimal order in $\mathcal{V}_\theta(B_0Z; q)$ not centralized by Z. Then $[Q_0, Z] = Q_0 \neq 1$. Since B_0 centralizes Z, the minimality of Q_0 also implies that Q_0 centralizes some involution b_0 of B_0. Setting $C = C(b_0)$, we have $\langle Q_0, U, A\rangle \subseteq C$, so $\langle Q_0, U\rangle \subseteq \theta(C)$. Since $[Q_0, Z] = Q_0$ and $\theta(C) \in \mathcal{V}_\theta(B_0)$, Lemma 3.17 yields that $Q_0 \subseteq O_{p'}(\theta(C))$. It follows from this that Z does not centralize any Sylow q subgroup of $O_{p'}(\theta(C))$ that it normalizes. But BU normalizes some Sylow q subgroup Q_1 of $O_{p'}(\theta(C))$. Since $Q_1 \in \mathcal{V}_\theta(BU; q)$, we have, therefore, shown that Z does not centralize some element of $\mathcal{V}_\theta(BU; q)$. We shall now contradict this fact.

Let $Q \in \mathcal{V}_\theta(BU; q)$ be of minimal order not centralized by Z. As usual, Lemma 3.15 implies that Q centralizes a subgroup U_0 of U of order p and that $[Q, Z] = Q \neq 1$. Setting $H = N(U_0)$, we have that $\langle Q, F\rangle \subseteq H$; therefore, $\langle Q, F\rangle \subseteq \theta(H)$. Moreover, $Q \subseteq O_{p'}(\theta(H))$ by another application of Lemma 3.17. But BF normalizes some Sylow q subgroup Q^* of $O_{p'}(\theta(H))$. Since $Q^* \in \mathcal{V}_\theta(BF; q)$, it follows, therefore, from Theorem 4.8 that F centralizes Q^*. In particular, Z centralizes Q^* and so Z centralizes every q subgroup of $O_{p'}(\theta(H))$ that it normalizes, contrary to the fact that $Q \subseteq O_{p'}(\theta(H))$ and Z does not centralize Q.

A slight modification of the first paragraph of the preceding proof will give our final result.

LEMMA 4.10. *If B_0 is any four subgroup of $C_A(Z)$ and Z centralizes every element of $\mathcal{V}_\theta(AZ; q)$, then Z centralizes every element of $\mathcal{V}_\theta(B_0Z; q)$.*

Proof. If Q_0 is chosen minimal to violate the conclusion, then, as usual, $Q_0 \neq 1$, $[Q_0, Z] = Q_0$ and Q_0 centralizes an involution b_0 of B_0. Setting $C = C(b_0)$, we have $A \subseteq C$ and $Q_0 \subseteq \theta(C)$, whence $Q_0 \subseteq O_{p'}(\theta(C))$ by Lemma 3.17. Since Z does not centralize Q_0, it follows that Z does not centralize any Sylow q subgroup of $O_{p'}(\theta(C))$ that it normalizes. However, AZ normalizes some Sylow q subgroup Q of $O_{p'}(\theta(C))$ and $Q \in И_\theta(AZ; q)$. But then Z centralizes Q by the hypothesis of the lemma. This contradiction completes the proof.

5. A SUBGROUP OF $Z(P)$

Let p be an odd prime for which $И_\theta^*(A; p)$ is nontrivial and let $P \in И_\theta^*(A; p)$. Also let q be a fixed odd prime distinct from p. In this section we shall establish the following basic result:

THEOREM 5.1. *One of the following two statements holds:*

(i) $C_P(Q_1) = 1$ *for every element* Q_1 *of* $И_\theta^*(AP; q)$; *or*

(ii) *There exists an* A*-invariant subgroup* Z *of* $Z(P)$ *of order* p *which centralizes every element of* $И_\theta(AZ; q)$.

We carry out the proof in a sequence of lemmas.

LEMMA 5.2. *If* P *centralizes every element of* $N_\theta(AP; q)$, *then the second alternative of the theorem holds.*

Proof. If $C_A(P)$ is cyclic, the hypothesis of the preceding section is satisfied and we can invoke Theorem 4.9. If Z and B_0 are as in that theorem, we conclude that Z centralizes every element of $И_\theta(B_0Z; q)$. But then, using Lemma 3.2, we see that Z centralizes every element of $И_\theta(AZ; q)$. Thus Z has the required property.

We can, therefore, assume that $C_A(P)$ contains a four subgroup B. In this case we argue that any A-invariant subgroup Z of $Z(P)$ of order p has the required properties. Let $Q \in И_\theta(AZ; q)$ be of minimal order not centralized by Z and suppose $Q \neq 1$. The minimality of Q implies that $[Q, Z] = Q$ and, as $B \subseteq Z(AZ)$, that Q centralizes some involution b of B. Setting $H = C(b)$ and using the fact that b centralizes P, it follows that $\langle Q, P \rangle \subseteq \theta(H)$ and that P is a Sylow p subgroup of $\theta(H)$. Lemma 3.17 also implies that $Q \subseteq O_{p'}(\theta(H))$ as $\theta(H) \in И_\theta(B)$ and $[Q, Z] = Q$. However, AP normalizes some Sylow q subgroup R of $O_{p'}(\theta(H))$ and so P centralizes R by the assumptions of the lemma. In particular, Z centralizes R and it follows that Z centralizes Q, a contradiction. We conclude that Z centralizes every element of $И_\theta(AZ; q)$.

We may, therefore, assume henceforth that P does not centralize every element of $И_\theta(AP; q)$. In particular, this implies that $1 \notin И_\theta^*(AP; q)$. If $C_P(Q_1) = 1$ for every element Q_1 of $И_\theta^*(AP; q)$, then the first alternative of the theorem holds, so we can also assume that this is not the case. Thus, we can choose Q_1 in $И_\theta^*(AP; q)$ so that $Q_1 \neq 1$ and $P_1 = C_P(Q_1) \neq 1$. Since

P_1 centralizes Q_1 and is normal in P, it is normal in PQ_1, so $P_1 \subseteq O_p(PQ_1)$. On the other hand, $O_p(PQ_1)$ clearly centralizes Q_1, so $P_1 = O_p(PQ_1)$. Since P_1 is normal in P, $P_1 \cap Z(P) \neq 1$ and, therefore, $P_1 \cap Z(P)$ contains an A-invariant subgroup Z of order p. We shall show that Z has the required properties. We set $H = N(Q_1)$ and fix all this notation.

LEMMA 5.3. *If Z centralizes the element R of $\mathsf{V}_\theta(AZ; q)$, then $R^x \subseteq \theta(H)$ for some element x in $C(AZ)$.*

Proof. Setting $C = AC(Z)$, we have $\langle A, R, Q_1, P\rangle \subseteq C$ and so $\langle R, Q_1, P\rangle \subseteq \theta(C)$ with P an A-invariant Sylow p subgroup of $\theta(C)$. Since $[Q_1, P]$ is a p' group, it follows from the p constraint of $\theta(C)$ that $Q_1 \subseteq O_{p'}(\theta(C))$. Since AP normalizes some Sylow q subgroup of $O_{p'}(\theta(C))$ containing Q_1, we conclude from the fact that $Q_1 \in \mathsf{V}_\theta{}^*(AP; q)$ that Q_1 itself must be a Sylow q subgroup of $O_{p'}(\theta(C))$. Hence, if Q^* is an A-invariant Sylow q subgroup of $\theta(C)$ containing Q_1, we have $Q^* \cap O_{p'}(\theta(C)) = Q_1$ and consequently Q_1 is normal in Q^*. But as R is A-invariant, $R^x \subseteq Q^*$ for some x in $C_{\theta(C)}(A)$. Thus, R^x normalizes Q_1 and $x \in C(AZ)$. Since $H = N(Q_1)$, $R^x \subseteq H$. Moreover, $R^x \in \mathsf{V}_\theta(A)$ as $R \in \mathsf{V}_\theta(A)$ and x centralizes A. Hence, $R \subseteq \theta(H)$ and the lemma is proved.

We next prove

LEMMA 5.4. *If $R \in \mathsf{V}_\theta(AZ; q)$ and $C_R(Z) \neq 1$, then Z centralizes R.*

Proof. Assume false and choose R to violate the lemma in such a way that $R_0 = C_R(Z)$ has maximum order and subject to this condition, minimize the order of R. Then $R_0 \subset R$ as Z does not centralize R. Hence, $N_R(R_0) \supset R_0$ and so Z does not centralize $N_R(R_0)$. Thus $R = N_R(R_0)$ by the minimality of R and consequently R_0 is normal in R. Moreover, $R_0 \neq 1$.

In view of the preceding lemma, we can assume without loss that $R_0 \subseteq \theta(H)$. We shall argue that R_0 contains Q_1, so suppose false. Since $R_0 \subseteq \theta(H)$, R_0 normalizes Q_1 and, therefore, $R_1 = R_0 N_{Q_1}(R_0) \supset R_0$. Furthermore, Z centralizes R_1 as it centralizes both R_0 and Q_1. Setting $N = N(R_0)$, we have that $\langle A, R_1, R, Z\rangle \subseteq N$ and $\langle R_1, R, Z\rangle \subseteq \theta(N)$. In addition, $[R, Z] \subseteq O_{p'}(\theta(N))$ by Lemma 3.17 as $[R, Z, Z] = [R, Z]$ and $\theta(N) \in \mathsf{V}_\theta(B)$. But Z does not centralize $[R, Z]$ as it does not centralize R. This implies that Z does not centralize any AZ-invariant Sylow q subgroup of $O_{p'}(\theta(N))$. However, R_1 normalizes some AZ-invariant Sylow q subgroup Q^* of $O_{p'}(\theta(N))$. Setting $R^* = R_1 Q^*$, we have that $R^* \in \mathsf{V}_\theta(AZ; q)$, $C_{R^*}(Z) \supseteq R_1 \supset R_0$, and Z does not centralize R^*. This contradicts our maximal choice of $C_R(Z)$ and establishes the desired assertion.

Finally we consider the action of $R_0 \times Z$ on R. Application of Theorem 5.3.4 of [3] yields that Z does not centralize $W = C_R(R_0)$. But $Q_1 \subseteq R_0$ by the preceding argument, so $W \subseteq C(Q_1) \subseteq H$. Since W is A-invariant, $W \in \mathcal{V}_\theta(A)$ and hence $W \subseteq \theta(H)$. Another application of Lemma 3.17 implies that $[W, Z] \subseteq O_{p'}(\theta(H))$. However, Q_1 is a Sylow q subgroup of $O_{p'}(\theta(H))$ as $Q_1 \in \mathcal{V}_\theta^*(AP; q)$. But Z centralizes Q_1 and, therefore, centralizes every q subgroup of $O_{p'}(\theta(H))$ that it normalizes. In particular, Z centralizes $[W, Z]$ and thus, Z centralizes W. This contradiction completes the proof of the lemma.

We can now easily complete the proof of the theorem. Suppose it is false and choose R minimal violating its conclusion. Then $[R, Z] = R \neq 1$ and AZ acts irreducibly on $\bar{R} = R/\Phi(R)$. Setting $A_0 = C_A(Z)$, we have that $m(A_0) \geqslant 3$ and that $A_0 \subseteq Z(AZ)$. If $B = C_{A_0}(\bar{R})$, the irreducibility of AZ on $\bar{R}$ now yields that $m(A_0/B) \leqslant 1$. Thus, $m(B) \geqslant 2$ and so B is noncyclic. Furthermore, B centralizes R as it centralizes $\bar{R} = R/\Phi(R)$.

Let $b \in B^\#$ and set $C = C(b)$. We shall argue that $C_{\theta(C)}(Z)$ is a q'group. Indeed, if not, let Q_0 be a nontrivial A-invariant q subgroup of $\theta(C)$ centralized by Z. Since $RZ \subseteq C$, we have, as usual, that $R \subseteq O_{p'}(\theta(C))$. But Q_0 normalizes some AZ-invariant Sylow q subgroup Q^* of $O_{p'}(\theta(C))$ and Z does not centralize Q^* as it does not centralize R. Thus, Z does not centralize $R^* = Q_0Q^*$. However, $C_{R^*}(Z) \supseteq Q_0 \neq 1$ and $R^* \in \mathcal{V}_\theta(AZ; q)$, so the preceding lemma is contradicted. This proves the assertion.

The preceding argument applies for each b in $B^\#$. However, B is noncyclic and B normalizes Q_1, which is nontrivial, so $C_{Q_1}(b) \neq 1$ for some b in $B^\#$. For such a choice of b, $C_{Q_1}(b)$ would be a nontrivial q subgroup of $C_{\theta(C(b))}(Z)$ and so $C_{\theta(C(b))}(Z)$ would not be a q' group, contrary to what we have just shown. This completes the proof of Theorem 5.1.

6. $E_{p,q}(A)$- and $C_{p,q}(A)$-Theorems

As in [4], we say that G satisfies $E_{p,q}(A)$, p, q distinct odd primes if for some element P in $\mathcal{V}_\theta^*(A; p)$ and some element Q in $\mathcal{V}_\theta^*(A; q)$, PQ is a group. Morevover, we call PQ an $S_{p,q}(A)$ subgroup of G. In addition, if two $S_{p,q}(A)$ subgroups of G are always conjugate by an element of $\theta(C(A))$, we say that G satisfies $C_{p,q}(A)$.

In this section we prove that G satisfies both $E_{p,q}(A)$ and $C_{p,q}(A)$. To begin with, we let P and Q be arbitrary fixed elements of $\mathcal{V}_\theta^*(A; p)$ and $\mathcal{V}_\theta^*(A; q)$. In view of Theorem 3.3, we can clearly assume without loss that P and Q are each nontrivial. We first establish $E_{p,q}(A)$. We note that Theorem 5.1 applies equally well to the elements of $\mathcal{V}_\theta^*(AP; q)$ and to those of $\mathcal{V}_\theta^*(AQ; p)$.

We begin with two special cases:

Lemma 6.1. *If there exists a nontrivial element Q_1 in $\mathcal{V}_\theta^*(AP; q)$ such that $C_P(Q_1) = 1$, then G satisfies $E_{p,q}(A)$.*

Proof. Setting $H = N(Q_1)$, we have $AP \subseteq H$ and, as usual, P is a Sylow p subgroup of $\theta(H)$. Let Q_2 be an A-invariant Sylow q subgroup of $\theta(H)$ permutable with P. In view of Theorem 3.3, we can assume without loss that $Q_2 \subseteq Q$. We argue that $Q_2 = Q$, and this will suffice to establish the lemma.

Clearly $Q_1 \subseteq Q_2$. Moreover, $O_p(PQ_2) \subseteq P$ and centralizes Q_1. Since $C_P(Q_1) = 1$, it follows that $O_p(PQ_2) = 1$. Glauberman's ZJ-theorem (Theorem 8.2.11 of [3]) now yields that $Z(JQ_2))$ is normal in PQ_2. Hence, if $N = N(Z(J(Q_2)))$, we obtain that $\langle A, P, Q_2\rangle \subseteq N$. Furthermore, $\langle Q \cap N, P\rangle \subseteq \theta(N)$ and P is a Sylow p subgroup of $\theta(N)$. Since $[Q_1, P]$ is a p' group, it follows from the p constraint of $\theta(N)$ that $Q_1 \subseteq O_{p'}(\theta(N))$. But $Q_1 \in \mathcal{V}_\theta^*(AP; q)$ and consequently Q_1 is a Sylow q subgroup of $O_{p'}(\theta(N))$. Since $Q_1 \subseteq Q$, this implies that $Q_1 = Q \cap O_{p'}(\theta(N))$; hence, Q_1 is normal in $Q \cap N$. However, $Q_2 = N_Q(Q_1)$ by definition of Q_2. Thus $Q_2 = Q \cap N$ and so $Q_2 = N_Q(Z(J(Q_2)))$. We conclude at once from this that $Q = Q_2$.

Clearly Lemma 6.1 also holds with p and q interchanged.

LEMMA 6.2. *If P centralizes every element of $\mathcal{V}_\theta^*(AP; q)$ and Q centralizes every element of $\mathcal{V}_\theta^*(AQ; p)$, then G satisfies $E_{p,q}(A)$.*

Proof. In this case, the second alternative of Theorem 5.1 holds for both p and q. Let Z_p, Z_q denote the corresponding A-invariant subgroups of $Z(P)$, $Z(Q)$ of order p, q, respectively, whose existence is asserted in that theorem. Then $C_A(Z_p)$ and $C_A(Z_q)$ each have index at most 2 in A. Since $m(A) = 4$, it follows that their intersection is nontrivial. Hence, there exists an involution a of A which centralizes both Z_p and Z_q.

Setting $C = C(a)$, it follows from Lemma 3.5 that $P_1 = P \cap \theta(C)$ and $Q_1 = Q \cap \theta(C)$ are, respectively, a Sylow p subgroup and a Sylow q subgroup of $\theta(C)$. In view of Theorem 3.3, we can assume without loss that P_1, Q_1 are permutable. We set $D = P_1Q_1$. By Theorem 6.1, Z_p centralizes every element of $\mathcal{V}_\theta(AZ_p; q)$ and so Z_p centralizes $O_q(D)$. But $Z_p \subseteq Z(P_1)$ as $P_1 \subseteq P$ and $Z_p \subseteq Z(P)$, so Z_p also centralizes $O_p(D)$. Thus,

$$Z_p \subseteq C_D(F(D)) \subseteq F(D)$$

as $F(D) = O_q(D) \times O_p(D)$ and D is solvable. Similarly we obtain that $Z_q \subseteq F(D)$. Hence, $Z_p \subseteq O_p(D)$ and $Z_q \subseteq O_q(D)$ and consequently Z_p centralizes Z_q.

We next set $H = N(Z_q)$. Then $\langle Q, Z_p, A\rangle \subseteq H$ and, as usual, $\langle Q, Z_p\rangle \subseteq \theta(H)$ with Q an A-invariant Sylow q subgroup of $\theta(H)$. Again replacing Q by an appropriate conjugate, we can assume without loss that Q is permutable with an A-invariant Sylow p subgroup P^* of $\theta(H)$ with $Z_p \subseteq P^*$. We set $E = P^*Q$ and argue that $Z_p \subseteq O_p(E)$. Indeed, $Z_p \subseteq O_{q,p}(E)$ by Theorem 3.16 as $E \in \mathcal{V}_\theta(B)$ for any four subgroup B of $C_A(Z_p)$. Furthermore, Z_p centralizes $O_q(E)$ by Theorem 5.1. But now if $X = O_{q,p}(E)$, we clearly have $O_q(E) = O_q(X)$, so $Z_p \subseteq X$ and Z_p centralizes $O_q(X)$. Since $O_q(X)$ is a Sylow q subgroup of X, it follows from the q constraint of X that $Z_p \subseteq O_p(X)$. But $O_p(X)$, being characteristic in X, is normal in E, so $O_p(X) \subseteq O_p(E)$. Thus $Z_p \subseteq O_p(E)$, as asserted.

Finally by the assumption of the lemma, Q centralizes $O_p(E)$ and therefore Q centralizes Z_p. Setting $N = N(Z_p)$, we have $\langle A, P, Q\rangle \subseteq N$ and, as usual, $\langle P, Q\rangle \subseteq \theta(N)$ with P a Sylow p subgroup and Q a Sylow q subgroup of $\theta(N)$. Again replacing Q by an appropriate conjugate, we can assume without loss that P and Q are permutable. Thus PQ is an $S_{p,q}(A)$ subgroup of G and so G satisfies $E_{p,q}(A)$.

We can now easily prove $E_{p,q}(A)$ by essentially the same arguments as in [4].

THEOREM 6.3. *G satisfies $E_{p,q}(A)$.*

Proof. In view of Lemma 6.2, we may assume that, say, P does not centralize some element of $\mathcal{H}_\theta^*(AP; q)$, otherwise G satisfies $E_{p,q}(A)$. In particular, $\mathcal{H}_\theta^*(P; q)$ is thus nontrivial. Furthermore, because of Lemma 6.1, we may also assume that $C_P(Q_1) \neq 1$ for some element Q_1 in $\mathcal{H}_\theta^*(P; q)$, or again G satisfies $E_{p,q}(A)$. Since $\mathcal{H}_\theta^*(AP; q)$ is nontrivial, $Q_1 \neq 1$.

We set $H = N(Q_1)$, so that P is a Sylow p subgroup of $\theta(H)$. We let Q_2 be an A-invariant Sylow q subgroup of $\theta(H)$ permutable with P and, without loss of generality, we may assume that $Q_2 \subseteq Q$. If $O_p(Q_2P) = 1$, then $Q_1 = O_q(Q_2P)$ inasmuch as $Q_1 \subseteq O_q(Q_2P)$ and $Q_1 \in \mathcal{H}_\theta^*(AP; q)$. But then $C_P(Q_1) = 1$ by the q constraint of Q_2P, contrary to our choice of Q_1. Hence, $P_1 = O_p(Q_2P) \neq 1$.

Likewise we may assume that the second alternative of Theorem 5.1 holds with p and q interchanged, or again G satisfies $E_{p,q}(A)$. In this case, Theorem 5.1 implies that $Z(Q)$ possesses an A-invariant subgroup Z_q of order q which centralizes every element of $\mathcal{H}_\theta(AZ_q; p)$. The existence of such a subgroup Z_q is critical for the proof.

Setting $K = N(P_1)$, the argument of Lemma 5.3 of [4] applies with no essential changes and yields the following: Q_2P is a Hall $\{p, q\}$-subgroup of $\theta(K)$ and P_1 is a Sylow p subgroup of $O_{q'}(\theta(K))$.

The next step in the proof is to establish a direct analogue of Lemma 5.5 of [4]: Namely, $\mathcal{H}_\theta(A)$ possesses a $\{p, q\}$-subgroup which contains both Q_2P_1 and an element of $\mathcal{H}_\theta^*(A; q)$. To prove this assertion, observe that $Z(Q) \subseteq H = N(Q_1)$ and so $Z(Q) \subseteq Q \cap \theta(H) = Q_2$. In particular, $Z_q \subseteq Q_2$. Since Q_2 normalizes P_1, it follows that $P_1 \in \mathcal{H}_\theta(AZ_q; p)$. Hence, Z_q centralizes P_1 by our choice of Z_q. Setting $M = N(Z_q)$, we see that $\langle P_1, Q, A\rangle \subseteq M$, whence $\langle P_1, Q\rangle \subseteq \theta(M)$ and Q is a Sylow q subgroup of $\theta(M)$. But now we see that an A-invariant Hall $\{p, q\}$ subgroup of $\theta(M)$ containing Q_2P_1 has the required properties.

Once we have this conclusion, we can repeat the proof of Theorem 5.6 of [4] without change to conclude that an A-invariant Hall $\{p, q\}$ subgroup of $\theta(N(P_1))$ is, in fact, an $S_{p,q}(A)$ subgroup of G. Thus G satisfies $E_{p,q}(A)$ in all cases.

We turn now to $C_{p,q}(A)$ and first prove

LEMMA 6.4. *If R is an $S_{p,q}(A)$ subgroup of G containing P, then*

$$O_q(R) \in \mathcal{U}_\theta^*(AP; q).$$

Proof. We reason as in Lemma 5.7 of [4] reducing first to the case that $O_q(R) \neq 1$. If $\mathcal{U}_\theta(AP; q)$ is trivial, the lemma is clear, so we may assume there exists $W \neq 1$ in $\mathcal{U}_\theta(AP; q)$. Since $m(A) = 4$, $W_0 = C_W(T) \neq 1$ for some four subgroup T of A and the reduction to the case $O_q(R) \neq 1$ follows exactly as in Lemma 5.7 of [4]. Setting $E = O_q(R)$ and $N = N(E)$, we conclude now by the same argument as in Lemma 5.7 of [4] that $O_q(R) \in \mathcal{U}_\theta^*(AP; q)$.

We now prove

THEOREM 6.5. G satisfies $C_{p,q}(A)$.

Proof. We follow the proof of Theorem 5.8 of [4], reducing at once to the case that, say, $E = O_q(R) \neq 1$. By Lemma 6.4, also $E_1 = O_q(R_1) \neq 1$. Without loss we may assume that $R = PQ$ and that $P \subseteq R_1$. Then $R_1 = PQ^x$ for some element x in $\theta(C(A))$ by Theorem 3.3.

Suppose first that $C_P(E) = C_P(E_1) = 1$. Application of Glauberman's ZJ-theorem then yields that $N = (Z(J(Q)))$ contains both R and $R_2 = R_1^{x^{-1}}$. But then $\langle R, R_2\rangle \subseteq \theta(N)$ and each is a Hall $\{p, q\}$ subgroup of $\theta(N)$. It follows that $R = R_2{}^y$ for some element y in $C_{\theta(N)}(A)$. Setting $u = x^{-1}y$, we conclude that $R = R_1{}^u$ and $u \in \theta(C(A))$.

Assume next that, say, $C_P(E) \neq 1$, in which case the second alternative of Theorem 5.1 holds and yields the existence of an A-invariant subgroup Z_p of $Z(P)$ which centralizes every element of $\mathcal{U}_\theta(AZ_p; q)$. In particular, Z_p centralizes both E and E_1. Setting $M = N(Z_p)$, it follows that $\langle E, E_1, P\rangle \subseteq \theta(M)$ and that P is a Sylow p subgroup of $\theta(M)$. Since $[E, P]$ and $[E_1, P]$ are p' groups, the p constraint of $\theta(M)$ implies that $\langle E, E_1\rangle \subseteq O_{p'}(\theta(M))$. But E, $E_1 \in \mathcal{U}_\theta^*(AP; q)$ by Lemma 6.4 and consequently each is a Sylow q subgroup of $O_{p'}(\theta(M))$. It follows at once that $E_1{}^v = E$ for some element v in $\theta(C(A))$. Now the final paragraph of the proof of Theorem 5.8 of [4] shows that R and R_1 are conjugate by an element of $\theta(C(A))$ and we conclude that G satisfies $C_{p,q}(A)$.

7. Proof of Theorem B

We can now complete the proof of Theorem B by essentially the same argument as given in Section 6 of [4]. As in that section, we say that G satisfies $E_\tau(A)$ for any set of primes τ provided $\mathcal{U}_\theta(A)$ possesses a τ subgroup K which contains an element of $\mathcal{U}_\theta^*(A; p)$ for each prime p in τ. Such a subgroup K is again called an $S_\tau(A)$ subgroup of G. Moreover, we say that G satisfies $C_\tau(A)$ if any two $S_\tau(A)$ subgroups of G are conjugate by an element of $\theta(C(A))$. Finally σ once again denotes the set of odd primes p such that $\mathcal{U}_\theta(A; p)$ is nontrivial.

As in Section 6 of [4], to prove Theorem B, it sill suffice to show that G satisfies $E_\sigma(A)$. First of all, the proof of Theorem 6.1 of [4] carries over without change and shows that G satisfies $C_\tau(A)$ for any subset τ of σ. The proof of Theorem 6.2 of [4] also carries over, with very minor modifications, to yield the desired conclusion that G satisfies $E_\sigma(A)$. We shall now describe these modifications. For clarity and simplicity, we use the identical notation as in Theorem 6.2 of [4].

The first part of the proof remains unchanged and we reach the situation in which $W \neq 1$. Here $W = O_{p_1'}(K_2)\, O_{p_1'}(K_3)$ with p_1, K_2, K_3 as in Theorem 6.2 of [4]. The AP_1-invariant Sylow p_i subgroups W_i of W are again elements of $\mathcal{U}_\theta^*(AP; p_i)$, $2 \leqslant i \leqslant n$. Setting $M = N(W)$ and $N_1 = N(Y(P_1))$, where $Y(P_1)$ denotes, as in Section 4 of [4], Glauberman's characteristic subgroup $ZJ^*(P_1)$ of P_1, the key point in the balance of the proof is to demonstrate that M contains an A-invariant Sylow p_i subgroup of $\theta(N_1)$ for each i, $2 \leqslant i \leqslant n$. Once this is established, the same proof as in Theorem 6.2 of [4] will show that $\theta(M)$ contains an $S_\tau(A)$ subgroup of G, so G satisfies $E_\tau(A)$ for the given subset τ or σ.

We again have that E_1 is normal in P_1A, $E_1 \neq 1$, and E_1 centralizes W, and we again let Z_1 denote a minimal A-invariant subgroup of $Z(P_1) \cap E_1$. Then Z_1 has order p and the proof of Theorem 5.1 shows that Z_1 centralizes every element of $\mathcal{U}_\theta(AZ_1; p_i)$, $2 \leqslant i \leqslant n$. Thus the hypotheses of Theorems 4.11–4.14 of [4] are again satisfied for each p_i, $2 \leqslant i \leqslant n$. Moreover, one checks that the proofs of these theorems remain valid for any noncyclic subgroup B of $A_1 = C_A(Z_1)$. In the present case, we use Lemma 4.10 to obtain the critical conclusion that Z_1 centralizes every element of $\mathcal{U}_\theta(BZ_1; p_i)$, $2 \leqslant i \leqslant n$.

We let $\mathscr{B}$ denote the set of noncyclic subgroups of A_1 and use the preceding results to conclude as in [4] that $C_M(B)$ contains a Sylow p_i subgroup of $C_{\theta(N_1)}(B)$ for each B in $\mathscr{B}$. But $m(A_1) \geqslant 3$ by Lemma 3.14(i). It follows, therefore, from the Wielandt formula as in [4] that M contains an A-invariant Sylow p_i subgroup of $\theta(N_1)$, $2 \leqslant i \leqslant n$, as required.

References

1. W. Feit and J. G. Thompson, Solvability of groups of odd order, *Pacific J. Math.* **13** (1963), 775–1029.
2. G. Glauberman, A characteristic subgroup of a p-stable group. *Canad. J. Math.* **20** (1968), 1101–1135.
3. D. Gorenstein, "Finite Groups," Harper and Row, New York, 1968.
4. D. Gorenstein, On the centralizers of involutions in finite groups, *J. Algebra* **11** (1969), 243–277.
5. D. Gorenstein, The flatness of signalizer functors in finite groups, *J. Algebra* **13** (1969), 509–512.